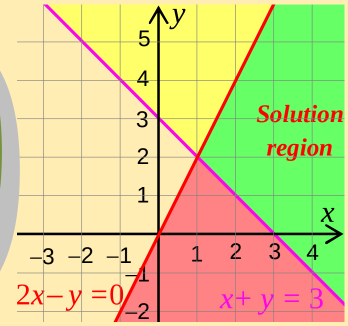
Unit 10



INTRODUCTION TO LINEAR PROGRAMMING

Unit Outcomes:

After completing this unit, you should be able to:

- *identify regions of inequality graphs.*
- *create real life examples of linear programming problems using inequalities and solve them.*

Main Contents:

10.1 REVISION ON LINEAR GRAPHS

10.2 GRAPHICAL SOLUTIONS OF SYSTEMS OF LINEAR INEQUALITIES

10.3 MAXIMUM AND MINIMUM VALUES

10.4 REAL LIFE LINEAR PROGRAMMING PROBLEMS

Key terms

Summary

Review Exercises

INTRODUCTION

MANY REAL LIFE PROBLEMS INVOLVE FINDING THE OPTIMUM (MAXIMUM OR MINIMUM) VALU FUNCTION UNDER CERTAIN CONDITIONS. IN PARTICULAR, LINEAR PROGRAMMING IS A MATHEMATICS THAT DEALS WITH THE PROBLEM OF FINDING THE MAXIMUM OR MINIMUM VA GIVEN LINEAR FUNCTION, KNOWN AS THE OBJECTIVE FUNCTION, SUBJECT TO CERTAIN CO EXPRESSED AS LINEAR INEQUALITIES KNOWN AS CONSTRAINTS. THE OBJECTIVE FUNCTION PROFIT, COST, PRODUCTION CAPACITY OR ANY OTHER MEASURE OF EFFECTIVENESS, WHIC OBTAINED IN THE BEST POSSIBLE OR OPTIMAL MANNER. THE CONSTRAINTS MAY BE IMPO DIFFERENT RESOURCE LIMITATIONS SUCH AS MARKET DEMAND, LABOUR TIME, PRODUCTIO ETC.

HISTORICAL NOTE

Leonid Vitalevich Kantorovich (1912-1986)

A Soviet Mathematician, and Economist, received his doctorate in 1930 at the age of eighteen. One of his most fundamental works on economics was The Best Use of Economic Resources (1959). Kantorovich pioneered the technique of linear programming as a tool of economic planning, having developed a linear programming model in 1939. He was a joint winner of the 1975 Nobel Prize for economics for his work on the optimal allocation of scarce resources.

OPENING PROBLEM

A MAN WANTS TO FENCE A PLOT OF LAND IN THE SHAPE OF A TRIANGLE WHOSE VERTICES POINTS A (4, 1), B (2, 5) AND C (–1, 0).

I IDENTIFY THIS REGION IN THE xy-PLANE;

II FIND THE EQUATION OF THE LINES THAT PASS THROUGH THE SIDES OF THIS REGION;

III EXPRESS THE REGION BOUNDED BY THE FENCES USING IN EQUALITIES.

10.1 REVISION ON LINEAR GRAPHS

GIVEN A NON HORIZONTAL LINE ℓ ON THE xy-COORDINATE PLANE, IT INTERSECTS THE x-AXIS AT EXACTLY ONE POINT. THE ANGLE θ MEASURED FROM THE x-AXIS TO ℓ IN THE COUNTER CLOCKWISE DIRECTION IS CALLED THE **inclination** OF THE LINE ℓ $(0 \le \theta < 180^{\circ})$.

IN ORDER TO DETERMINE THE EQUATION OF ℓ, WE PICK TWO POINTS P(x_1, y_1) AND Q(x_2, y_2) ON ℓ AS SHOWN IN FIGURE 10.1. THEN WE DEFINE THE SLOPE OF ℓ BY

$$m = \frac{\text{RISE}}{\text{RUN}} = \frac{y_2 - y_1}{x_2 - x_1}, \text{ FOR } x_1 \neq x_2.$$

$$\text{SINCE } \tan\theta = \frac{\text{OPPOSITE SIDE}}{\text{ADJACENT SIDE}} = \frac{y_2 - y_1}{x_2 - x_1}, \text{ WE HAVE } m = \tan\theta$$

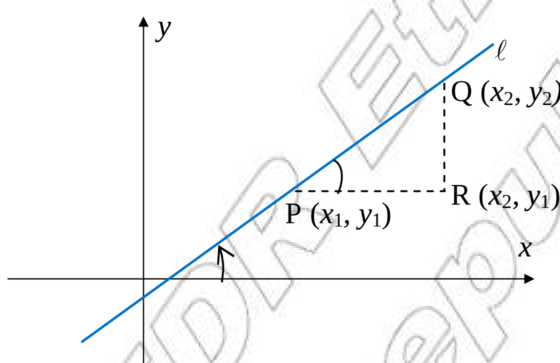


Figure 10.1

Example 1 THE SLOPE OF A LINE ℓ PASSING THROUGH THE POINTS P (3, –2) AND Q (–1, 3) IS GIVEN BY $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-(-2)}{-1-3} = -\dfrac{5}{4}$.

TWO NON-VERTICAL LINES ℓ_1 AND ℓ_2 WITH SLOPES m_1 AND m_2, RESPECTIVELY, ARE **parallel** IF AND ONLY IF THEY HAVE THE SAME SLOPE, I.E, $m_1 = m_2$.

ACTIVITY 10.1

A FIND THE VALUE OF k SO THAT THE LINE PASSING THROUGH THE POINTS P (1, –2) AND Q (k, 3) HAS SLOPE 5.

B VERIFY THAT THE LINE ℓ_1 THROUGH THE POINTS A (1, 1) AND B (–2, 3) IS PARALLEL TO THE LINE ℓ_2 THROUGH THE POINTS C (3, 2) AND D (–3, 6).

AN **equation** OF A LINE ℓ IS AN EQUATION IN TWO VARIABLES x AND y SUCH THAT A POINT P (x, y) IS ON ℓ IF AND ONLY IF x AND y SATISFY THE EQUATION.

RECALL THAT IF A LINE ℓ HAS SLOPE m AND PASSES THROUGH A POINT (x_1, y_1), THEN THE POINT-SLOPE FORM OF EQUATION OF ℓ IS GIVEN BY

$$y - y_1 = m(x - x_1)$$

IF THE LINE PASSES THROUGH (0, 0), ITS EQUATION IS $y = mx$.

Example 2 THE EQUATION OF THE LINE PASSING THROUGH $(-2, 3)$ WITH SLOPE $m = 2$ IS GIVEN BY $y - 3 = 2(x - (-2)) = 2(x + 2) = 2x + 4$ OR $y = 2x + 7$.

IF THE y-INTERCEPT OF A LINE WITH SLOPE m IS $(0, b)$, THEN ITS EQUATION IN THE SLOPE-INTERCEPT FORM IS

$$y = mx + b$$

Example 3 THE EQUATION OF A LINE ℓ WITH SLOPE $\frac{1}{2}$ AND y-INTERCEPT -3 IS GIVEN BY

$$y = \frac{1}{2}x - 3 \quad \text{OR} \quad 2y = x - 6$$

TO SKETCH THE GRAPH OF THIS LINE IN THE xy-PLANE, WE NEED TO PLOT TWO POINTS. ONE OF THESE IS THE y-INTERCEPT $(0, -3)$. TO GET A SECOND POINT, TAKE $x = 2$. THEN $y = -2$, SO THAT THE POINT $(2, -2)$ IS ON THE LINE ℓ.

USING THESE TWO POINTS, THE LINE ℓ CAN BE DRAWN AS SHOWN IN FIGURE 10.2. IF A LINE ℓ_1 HAS THE SAME SLOPE $\frac{1}{2}$ (I.E., ℓ_1 IS PARALLEL TO ℓ) AND HAS y-INTERCEPT -1, ITS EQUATION IS $y = \frac{1}{2}x - 1$ OR $2y = x - 2$.

ITS GRAPH IS SHOWN IN FIGURE 10.2.

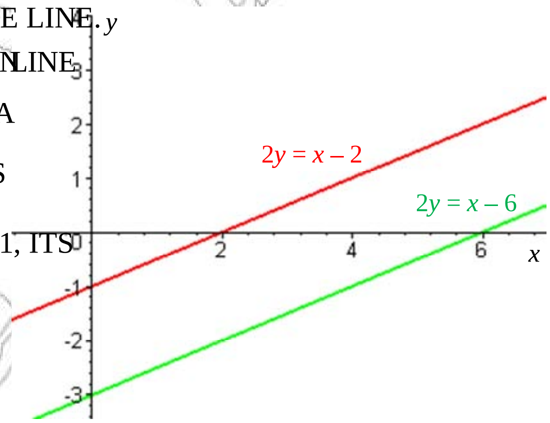


Figure 10.2

ANY EQUATION OF A LINE CAN BE REDUCED TO THE FORM $ax + by = c$ WHERE $a, b, c \in \mathbb{R}$ WITH $a \neq 0$ OR $b \neq 0$.

Example 4 IF A LINE ℓ PASSES THROUGH P $(1, -3)$ AND Q $(2, 2)$, THEN ITS SLOPE IS

$$m = \frac{2+3}{2-1} = 5$$

ITS EQUATION IN SLOPE-INTERCEPT FORM IS

$$y - 2 = 5(x - 2) = 5x - 10 \quad \text{OR} \quad y = 5x - 8 \text{ (SLOPE 5, } y\text{-INTERCEPT } (0, -8))$$

THIS CAN BE WRITTEN IN THE FORM $5x - y = 8$, WITH $a = 5$, $b = -1$ AND $c = 8$.

Note:

1. AN EQUATION OF A VERTICAL LINE PASSING THROUGH THE POINT (h, k) IS GIVEN BY $x = h$. A VERTICAL LINE HAS NO SLOPE.
2. AN EQUATION OF A HORIZONTAL LINE PASSING THROUGH THE POINT (h, k) IS GIVEN BY $y = k$. A HORIZONTAL LINE HAS ZERO SLOPE.
3. TWO LINES ARE PERPENDICULAR, IF AND ONLY IF THEIR SLOPES ARE NEGATIVE RECIPROCALS OF EACH OTHER. THAT IS, IF ℓ_1 HAS SLOPE m_1 AND ℓ_2 HAS SLOPE m_2, THEN ℓ_1 IS PERPENDICULAR TO ℓ_2 IF AND ONLY IF $m_1 m_2 = -1$.

Exercise 10.1

1. DETERMINE THE EQUATION OF THE LINE
   - **A** THAT HAS SLOPE 4 AND PASSES THROUGH P $(-1, 3)$.
   - **B** THAT PASSES THROUGH THE POINTS P $(1, 2)$ AND Q $(-4, 1)$.
   - **C** WHOSE SLOPE IS -2 WITH y-INTERCEPT $(0, 5)$.
2. DETERMINE THE VALUE OF k SO THAT THE LINE WITH EQUATION $4x + ky = 8$ IS PARALLEL TO THE LINE WITH EQUATION $x + 2y = 0$.
3. DRAW THE GRAPHS OF THE FOLLOWING LINES USING THE COORDINATE AXES.
   - **A** $y = 2x - 1$ **B** $y = 2x + 3$ **C** $3x - 2y = 4$

10.2 GRAPHICAL SOLUTIONS OF SYSTEMS OF LINEAR INEQUALITIES

IN THIS SECTION, YOU USE GRAPHS TO DETERMINE THE SOLUTION SET OF A SYSTEM OF INEQUALITIES IN TWO VARIABLES.

EVERY LINE $ax + by = c$ IN THE PLANE DIVIDES THE PLANE INTO TWO REGIONS, ONE ON EACH SIDE OF THE LINE. EACH OF THESE REGIONS IS CALLED A **half plane**. A VERTICAL LINE $x = a$ DIVIDES THE PLANE INTO LEFT AND RIGHT HALF PLANES. A POINT (x, y) IS TO THE LEFT OF THE LINE $x = a$, IF AND ONLY IF $x < a$. HENCE THE GRAPH OF THE INEQUALITY $x < a$ IS THE HALF PLANE LYING TO THE LEFT OF THE LINE $x = a$. SIMILARLY, THE GRAPH OF THE INEQUALITY $x > a$ IS THE HALF PLANE LYING TO THE RIGHT OF THE LINE $x = a$.

Example 1 LET ℓ BE THE VERTICAL LINE $x = 2$.

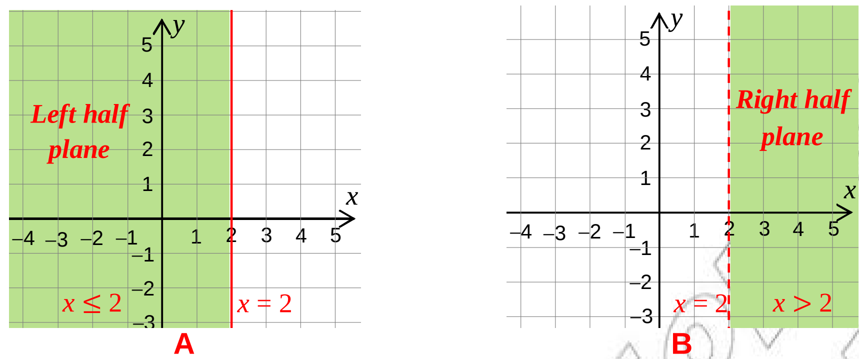


Figure 10.3

OBSERVE THAT THE LEFT HALF PLANE A: $x \le 2$ CONTAINS THE POINTS ON THE LINE $x = 2$ AND HENCE THE LINE IS A BOLD (UNBROKEN) LINE; WHEREAS THE RIGHT HALF PLANE B: $x > 2$ DOES NOT INCLUDE THE POINTS ON THE LINE $x = 2$ (BROKEN LINE).

A NON-VERTICAL LINE DIVIDES THE PLANE INTO TWO REGIONS WHICH CAN BE CALLED **upper and lower half planes.**

Example 2 CONSIDER THE GRAPH OF THE LINEAR EQUATION $2x - y = 3$ AND THE RELATED LINEAR INEQUALITIES $2x - y \ge 3$ AND $2x - y < 3$. FIRST GRAPH THE LINE $2x - y = 3$ BY PLOTTING TWO POINTS ON THE LINE. TO IDENTIFY WHICH HALF PLANE BELON WHICH INEQUALITY, TEST A POINT THAT DOES NOT LIE ON THE LINE (USUALLY THE

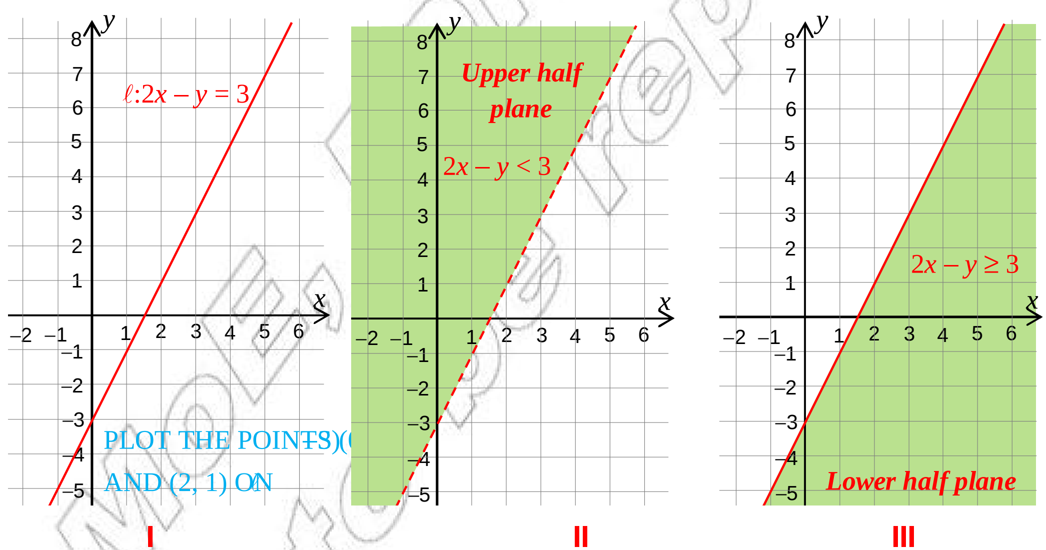


Figure 10.4

TEST (0, 0); 2(0) – 0 = 0 < 3

OBSERVE THE BROKEN LINE FOR $2x - y < 3$ AND SOLID LINE FOR $2x - y \ge 3$.

ACTIVITY 10.2

DRAW THE GRAPH OF EACH OF THE FOLLOWING INEQUALITIES:

A	$x \geq 0$	**B**	$y < -1$	**C**	$y \geq 3x$
D	$x > 2y$	**E**	$4x + y \geq 1$	**F**	$-x + 3y < 2$

A **system of linear inequalities** IS A COLLECTION OF TWO OR MORE LINEAR INEQUALITIES TO B SOLVED SIMULTANEOUSLY. A **graphical solution** OF A SYSTEM OF LINEAR INEQUALITIES IS THE GRAPH OF ALL ORDERED PAIRS (x, y) THAT SATISFY ALL THE INEQUALITIES. SUCH A GRAPH IS CALLE THE **solution region** (OR **feasible region**).

Example 3 FIND A GRAPHICAL SOLUTION TO THE SYSTEM OF LINEAR INEQUALITIES.

$$\left.\begin{aligned} x + y &\geq 3 \\ 2x - y &\geq 0 \end{aligned}\right\}$$

Solution FIRST DRAW THE LINES $x + y = 3$ AND $2x - y = 0$ BY PLOTTING TWO POINTS FOR EACH LINE. THEN SHADE THE REGIONS FOR THE TWO INEQUALITIES.

THE SOLUTION REGION IS THE INTERSECTION OF THE TWO REGIONS. TO FIND THE POINT INTERSECTION OF THE TWO LINES, SOLVE

$$\left.\begin{aligned} x + y &= 3 \\ 2x - y &= 0 \end{aligned}\right\}$$ SIMULTANEOUSLY, TO GET THE POINT (1, 2).

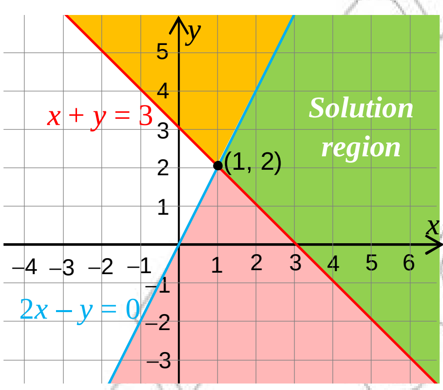


Figure 10.5

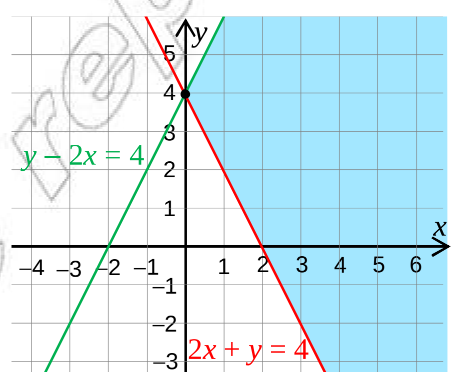


Figure 10.6

Example 4 DRAW THE SOLUTION REGION OF THE SYSTEM OF LINEAR INEQUALITIES.

$$\left.\begin{aligned} y - 2x &\leq 4 \\ 2x + y &\geq 4 \end{aligned}\right\}$$

Solution DRAW THE TWO LINES ℓ_1: $y - 2x = 4$ AND ℓ_2: $2x + y = 4$ AND IDENTIFY THEIR POINT OF INTERSECTION $(0, 4)$. THE SOLUTION REGION, WHICH IS THE INTERSECTION OF THE TWO HALF PLANES, IS SHADED IN FIGURE 10.6

Definition 10.1

A POINT OF INTERSECTION OF TWO OR MORE BOUNDARY LINES OF A SOLUTION REGION IS **vertex** (OR A **corner point**) OF THE REGION.

Example 5 SOLVE THE FOLLOWING SYSTEM OF LINEAR INEQUALITIES.

$$\left.\begin{aligned} 2x + y &\le 22 \\ x + y &\le 13 \\ 2x + 5y &\le 50 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}\right\}$$

Solution THE LAST TWO INEQUALITIES $x \ge 0$ AND $y \ge 0$ ARE KNOWN AS NON-NEGATIVE INEQUALITIES (OR NON-NEGATIVE REQUIREMENTS). THEY INDICATE THAT SOLUTION REGION IS IN THE FIRST QUADRANT OF THE PLANE.

DRAW THE LINES

$\ell_1 : 2x + y = 22$, $\ell_2 : x + y = 13$ AND $\ell_3 : 2x + 5y = 50$

TO DETERMINE THE SOLUTION REGION TEST THE POINT O (0, 0) WHICH IS NOT IN ANY OF THESE 3 LINES, AND FIND THE INTERSECTION OF ALL HALF PLANES TO GET THE SHADED FIGURE 10.7.

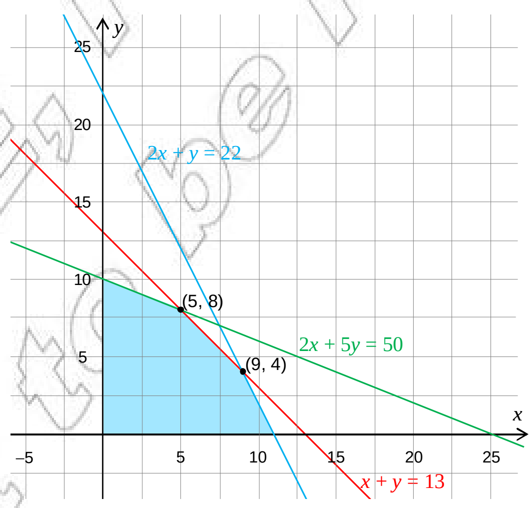


Figure 10.7

THIS SOLUTION REGION HAS FIVE CORNER POINTS. THE VERTICES O (0, 0), P (0, 10) AN Q (11, 0) CAN BE EASILY DETERMINED. TO FIND THE OTHER TWO VERTICES R AND S SO SIMULTANEOUSLY THE FOLLOWING TWO PAIRS OF EQUATIONS:

$$\left.\begin{aligned}\ell_1 &: 2x+y=22\\ \ell_2 &: x+y=13\end{aligned}\right\} \text{ AND } \left.\begin{aligned}\ell_2 &: x+y=13\\ \ell_3 &: 2x+5y=50\end{aligned}\right\}$$

TO GET S (9, 4) TO GET R (5, 8)

OBSERVE THAT THE POINT OF INTERSECTION OF ℓ_1 AND ℓ_3 IS NOT A CORNER POINT OF THE SOLUTION REGION.

Definition 10.2

A SOLUTION REGION OF A SYSTEM OF LINEAR INEQUALITIES IS SAID TO BE **bounded**, IF IT CAN BE ENCLOSED BY A RECTANGLE, OTHERWISE IT IS **unbounded**.

THUS THE SOLUTION REGION OF EXAMPLE 5 IS BOUNDED, WHILE THAT OF EXAMPLE 4 IS UNBOUNDED.

Exercise 10.2

FIND A GRAPHICAL SOLUTION FOR EACH OF THE FOLLOWING.

A	B	C	D
$x \geq 0$	$x - y \leq 2$	$x \geq 0$	$x, y \geq 0$
$y \geq 0$	$x + y \geq 2$	$y \geq 0$	$2x + 3y \leq 60$
$2x + 3y \leq 4$	$x + 2y \leq 8$	$x + y \geq 8$	$2x + y \leq 28$
	$x \leq 4$	$3x + 5y \geq 30$	$4x + y \leq 48$

10.3 MAXIMUM AND MINIMUM VALUES

Group Work 10.1

FIND TWO POSITIVE NUMBERS WHOSE SUM IS AT LEAS DIFFERENCE IS AT MOST 7, SUCH THAT THEIR PRODUCT IS

A MINIMUM B MAXIMUM

MANY APPLICATIONS IN BUSINESS AND ECONOMICS INVOLVE A PROCESS CALLED **optimization**, IN WHICH YOU ARE ASKED TO FIND THE MAXIMUM OR MINIMUM VALUE OF A QUANTITY. IN SECTION YOU WILL STUDY AN OPTIMIZATION STRATEGY CALLED **linear programming**.

Definition 10.3

SUPPOSE f IS A FUNCTION WITH DOMAIN $I = \{x \mid a \le x \le b\}$

I A NUMBER $M = f(c)$ FOR SOME c IN I IS CALLED THE **maximum** VALUE OF f ON I, IF

$M \ge f(x)$, FOR ALL x IN I.

II A NUMBER $m = f(d)$ FOR SOME d IN I IS CALLED THE **minimum** VALUE OF f ON I, IF

$m \le f(x)$, FOR ALL x IN I.

III A VALUE WHICH IS EITHER A MAXIMUM OR A MINIMUM IS CALLED AN **optimum** (OR **extremum**) VALUE OF f ON I.

MANY OPTIMIZATION PROBLEMS INVOLVE MAXIMIZING OR MINIMIZING A LINEAR FUN **(the objective function)** SUBJECT TO ONE OR MORE LINEAR EQUATIONS OR INEQUALIT **(constraints)**.

IN THIS SECTION, PROBLEMS WITH ONLY TWO VARIABLES ARE GOING TO BE CONSIDERED S PROBLEMS CAN EASILY BE SOLVED BY A GRAPHICAL METHOD.

Example 1 FIND THE VALUES OF x AND y WHICH WILL MAXIMIZE THE VALUE OF THE OBJECTIVE FUNCTION

$Z = f(x, y) = 2x + 5y$, SUBJECT TO THE LINEAR CONSTRAINTS:

$$x \ge 0$$

$$y \ge 0$$

$$3x + 2y \le 6$$

$$-2x + 4y \le 8$$

Solution: FIRST YOU SKETCH THE GRAPHICAL SOLUTION OF THE GIVEN CONSTRAINTS USING THE METHODS OF SECTION 10.2

THIS BOUNDED REGION S IS ALSO CALLED THE **region of feasible solution** OR **feasible region**.

ANY POINT IN THE INTERIOR OR ON THE BOUNDARY OF S SATISFIES ALL THE ABOVE CON

NEXT YOU FIND A POINT (x, y) OF THE FEASIBLE REGION THAT GIVES THE MAXIMUM VALUE OF THE OBJECTIVE FUNCTION Z. LET'S FIRST DRAW SOME LINES WHICH REPRESENT THE OB FUNCTION FOR VALUES OF Z = 0, 5, 10 AND 15; I.E., THE LINES

$2x + 5y = 0$ $\quad\quad 2x + 5y = 10$

$2x + 5y = 5$ $\quad\quad 2x + 5y = 15$

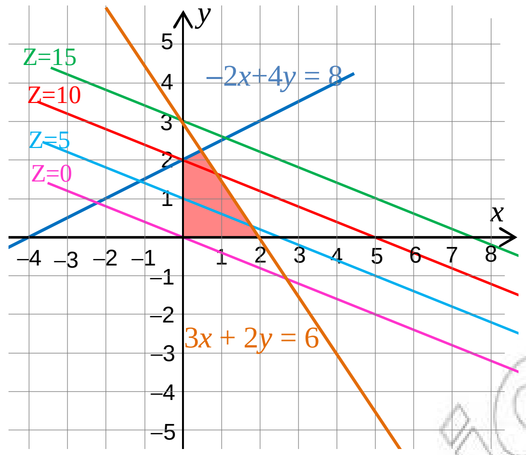


Figure 10.8

FROM FIGURE 10.8 YOU CAN OBSERVE THAT AS THE VALUE OF Z INCREASES, THE LINES A MOVING UPWARDS AND THE LINE FOR Z = 15 IS OUTSIDE THE FEASIBLE REGION. T MAXIMUM POSSIBLE VALUE OF Z WILL BE OBTAINED IF WE DRAW A LINE BETWEEN Z = AND Z = 15 PARALLEL TO THEM THAT JUST "TOUCHES" THE FEASIBLE REGION.

THIS OCCURS AT THE VERTEX (CORNER POINT) P WHICH IS THE POINT OF INTERSECTION OF

$$\left.\begin{aligned} 3x+2y&=6 \\ -2x+4y&=8 \end{aligned}\right\} \Rightarrow x = \frac{1}{2} \text{ AND } y = \frac{9}{4}$$

THE VALUE OF Z AT THIS POINT IS

$$Z = 2x + 5y = 2\left(\frac{1}{2}\right) + 5\left(\frac{9}{4}\right) = \frac{49}{4} = 12\frac{1}{4}$$

THUS THE MAXIMUM VALUE OF Z UNDER THE GIVEN CONDITIONS IS $Z = 12\frac{1}{4}$

AS A GENERALIZATION OF THIS EXAMPLE, WE STATE THE FOLLOWING:

Fundamental theorem of linear programming

Theorem 10.1

IF THE FEASIBLE REGION OF A LINEAR PROGRAMMING PROBLEM IS **non-empty** AND **bounded**, THEN THE OBJECTIVE FUNCTION ATTAINS BOTH A MAXIMUM AND A MINIMUM VALUE AND OCCUR AT CORNER POINTS OF THE FEASIBLE REGION. IF THE FEASIBLE REGION IS **unbounded**, THEN THE OBJECTIVE FUNCTION MAY OR MAY NOT ATTAIN A MAXIMUM OR MINIMUM VA HOWEVER, IF IT ATTAINS A MAXIMUM OR MINIMUM VALUE, IT DOES SO AT CORNER POINTS.

Steps to solve a linear programming problem by the graphical method

1. DRAW THE GRAPH OF THE FEASIBLE REGION.
2. COMPUTE THE COORDINATES OF THE CORNER POINTS.
3. SUBSTITUTE THE COORDINATES OF THE CORNER POINTS INTO THE OBJECTIVE FUNCTION TO SEE WHICH GIVES THE OPTIMAL VALUE.
4. IF THE FEASIBLE REGION IS UNBOUNDED, THIS METHOD IS MISLEADING: OPTIMAL SOLUTIONS ALWAYS EXIST WHEN THE FEASIBLE REGION IS BOUNDED, BUT MAY OR MAY NOT EXIS IT IS UNBOUNDED.

TO APPLY THIS TO Example 1, WE FIND THE VERTEX POINTS (0, 0), $\left(\frac{1}{2}, \frac{9}{4}\right)$, (2, 0) AND (0, 2)

AND TEST THEIR VALUES AS SHOWN IN THE FOLLOWING TABLE.

Vertex Point	Value of Z = 2x + 5y
(0, 0)	Z = 2(0) + 5 (0) = 0
(2, 0)	Z = 2 (2)+ 5 (0) = 4
$\left(\frac{1}{2}, \frac{9}{4}\right)$	$Z = 2\left(\frac{1}{2}\right) + 5\left(\frac{9}{4}\right) = \frac{49}{4}$
(0, 2)	Z = 2 (0) + 5 (2) = 10

COMPARING THE VALUES OF Z, YOU GET THE MAXIMUM VALUE OF $Z = \frac{49}{4}$ OBTAINED AT $\left(\frac{1}{2}, \frac{9}{4}\right)$.

WE ALSO HAVE THE MINIMUM VALUE $Z = 0$ AT (0, 0).

Example 2 SOLVE THE FOLLOWING LINEAR PROGRAMMING PROBLEM. FIND THE MAXIMUM VALUE OF THE OBJECTIVE FUNCTION $Z = 3x + 2y$, SUBJECT TO THE FOLLOWING CONSTRAINTS:

$$x \geq 0$$
$$y \geq 0$$
$$x + 2y \leq 4$$
$$x - y \leq 1$$

Solution: FROM THE CONSTRAINTS YOU SKETCH THE FEASIBLE REGION AS SHOWN IN FIGURE 10.9.

THE VERTICES OF THIS REGION ARE (0, 0), (1, 0), (2, 1) AND (0, 2).

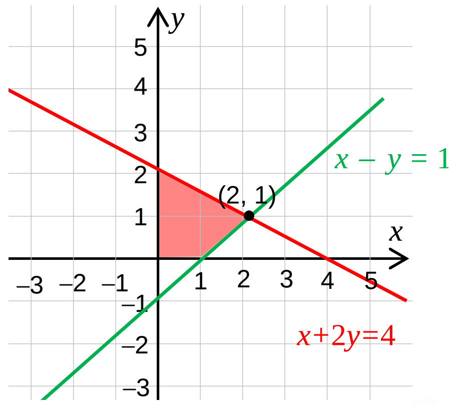


Figure 10.9

THEIR FUNCTIONAL VALUES AT Z ARE GIVEN IN THE FOLLOWING TABLE:

Vertex	Value of Z = 3x + 2y
(0, 0)	Z = 3 (0) + 2 (0) = 0
(1, 0)	Z = 3 (1) + 2 (0) = 3
(2, 1)	Z = 3 (2) + 2 (1) = 8
(0, 2)	Z = 3 (0) + 2 (2) = 4

THUS, THE MAXIMUM VALUE OF Z IS 8, AND OCCURS WHEN $x = 2$ AND $y = 1$.

ACTIVITY 10.3

1 IN EXAMPLE 2, TAKE SOME POINTS INSIDE THE REGION S A… THAT THEIR CORRESPONDING Z VALUES ARE LESS THAN 8.

2 FIND THE MAXIMUM AND MINIMUM VALUES OF

A OBJECTIVE FUNCTION: Z = 6x + 10y

Subject to: $x \geq 0$

$y \geq 0$

$2x + 5y \leq 10$

B OBJECTIVE FUNCTION: Z = 4x + y

Subject to: $x \geq 0$

$y \geq 0$

$x + 2y \leq 40$

$2x + 3y \leq 72$

Example 3 SOLVE THE FOLLOWING LINEAR PROGRAMMING PROBLEM.

FIND THE MAXIMUM VALUE OF $Z = 4x + 6y$ SUBJECT TO THE FOLLOWING CONSTRAINTS:

$$x \geq 0$$

$$y \geq 0$$

$$-x + y \leq 11$$

$$x + y \leq 27$$

$$2x + 5y \leq 90$$

Solution THE FEASIBLE REGION BOUNDED BY THE CONSTRAINTS IS SHOWN IN FIGURE 10.10. THE VERTICES OF THE FEASIBLE REGION ARE (0, 0), (27, 0), (15, 12), (5, 16) AND (0, 11).

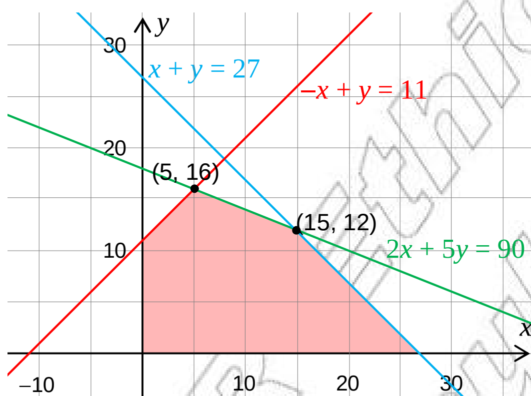


Figure 10.10

TESTING THE OBJECTIVE FUNCTION AT THE VERTICES GIVES

Vertex	Value of $Z = 2x + 4y$
(0, 0)	$Z = 4(0) + 6(0) = 0$
(27, 0)	$Z = 4(27) + 6(0) = 108$
(15, 12)	$Z = 4(15) + 6(12) = 132$
(5, 16)	$Z = 4(5) + 6(16) = 116$
(0, 11)	$Z = 4(0) + 6(11) = 66$

THUS THE MAXIMUM VALUE OF Z IS 132 WHEN $x = 15$ AND $y = 12$.

Example 4 FIND VALUES OF x AND y WHICH MINIMIZE THE VALUE OF THE OBJECTIVE FUNCTION

$Z = 2x + 4y$, SUBJECT TO: $x \geq 0$

$$y \geq 0$$

$$x + 2y \geq 10$$

$$3x + y \geq 10$$

Solution: FROM THE GIVEN CONSTRAINTS THE FEASIBLE REGION IS AS SHOWN IN FIGURE 10.11.

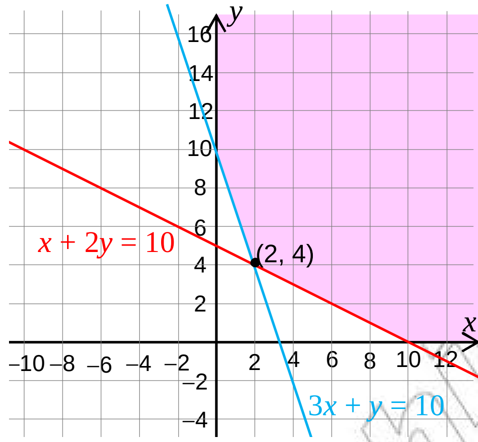


Figure 10.11

THIS REGION IS UNBOUNDED. THE VERTICES ARE AT (0, 10), (2, 4) AND (10, 0) WITH VALUES GIVEN BELOW.

Vertex	Value of Z
(0, 10)	2 (0) + 4 (10) = 40
(2, 4)	2 (2) + 4 (4) = 20
(10, 0)	2 (10) + 4(0) = 20

HERE VERTICES (2, 4) AND (10, 0) GIVE THE MINIMUM VALUE $Z = 20$ SO THAT THE SOLUTION IS NOT UNIQUE. IN FACT EVERY POINT ON THE LINE SEGMENT THROUGH (2, 4) AND (10, 0) GIVES SAME MINIMUM VALUE OF $Z = 20$.

FROM THIS EXAMPLE WE CAN OBSERVE THAT

- **I** AN OPTIMIZATION PROBLEM CAN HAVE INFINITE SOLUTIONS.
- **II** NOT ALL OPTIMIZATION PROBLEMS HAVE A SOLUTION, SINCE THE ABOVE PROBLEM NOT HAVE A MAXIMUM VALUE FOR Z.

Example 5 FIND VALUES OF x AND y THAT MAXIMIZE

$Z = x + 3y$, SUBJECT TO: $x + 3y \le 24$

$$x - y \le 7$$

$$y \le 6$$

$$x \ge 0$$

$$y \ge 0$$

Solution IN FIGURE 10.12 WE HAVE DRAWN THE FEASIBLE REGION OF THIS PROBLEM. SINCE IT IS BOUNDED, THE MAXIMUM VALUE OF Z IS ATTAINED AT ONE OF FIVE EXTR POINTS. THE VALUES OF THE OBJECTIVE FUNCTION AT THE FIVE EXTREME POINT GIVEN IN THE FOLLOWING TABLE.

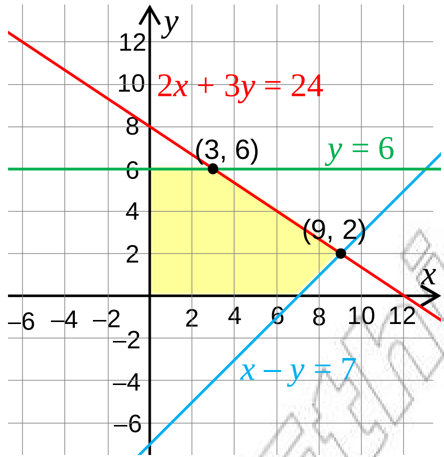


Figure 10.12

Corner point (x, y)	Value of $Z = x + 3y$
(0,6)	18
(3,6)	21
(9,2)	15
(7,0)	7
(0,0)	0

FROM THIS TABLE THE MAXIMUM VALUE OF Z IS 21, WHICH IS ATTAINED AT $x = 3$ AND $y = 6$.

Example 6 FIND VALUES OF x AND y THAT MINIMIZE

$Z = 2x - y$, SUBJECT TO: $x + 2y = 12$

$$2x - 3y \geq 0$$

$$x, y \geq 0$$

Solution: IN FIGURE 10.13 WE HAVE DRAWN THE FEASIBLE REGION OF THIS PROBLEM. BECAUSE ONE OF THE CONSTRAINTS IS AN EQUALITY CONSTRAINT, THE FE REGION IS A STRAIGHT LINE SEGMENT WITH TWO EXTREME POINTS. THE VALUES AT THE TWO EXTREME POINTS ARE GIVEN IN THE FOLLOWING TABLE.

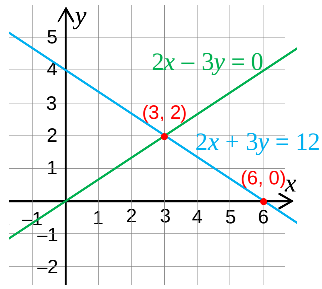


Figure 10.13

Extreme point (x, y)	Value of $Z=2x-y$
(3,2)	4
(6,0)	12

THUS THE MINIMUM VALUE OF $Z = 2x - y$ IS 4 ATTAINED AT (3, 2)

Example 7 MAXIMIZE $Z = 2x + 5y$ SUBJECT TO: $2x + y \geq 8$

$$-4x + y \leq 2$$

$$2x - 3y \leq 0$$

$$x, y \geq 0$$

Solution: THE FEASIBLE REGION IS ILLUSTRATED IN FIGURE 10.14. SINCE IT IS UNBOUNDED, WE ARE NOT ASSURED BY THEOREM 10.1 THAT THE OBJECTIVE FUNCTION ATTAINS A MAXIMUM VALUE. IN FACT, IT IS EASILY SEEN THAT SINCE THE FEASIBLE REGION CONTAINS POINTS FOR WHI[CH BOTH] x AND y ARE ARBITRARILY LARGE AND POSITIVE, THE OBJECTIVE FUNCTION CAN BE MADE A[RBITRARILY] LARGE AND POSITIVE. THIS PROBLEM HAS NO OPTIMAL SOLUTION. INSTEAD, WE SAY THE P[ROBLEM] HAS AN UNBOUNDED SOLUTION.

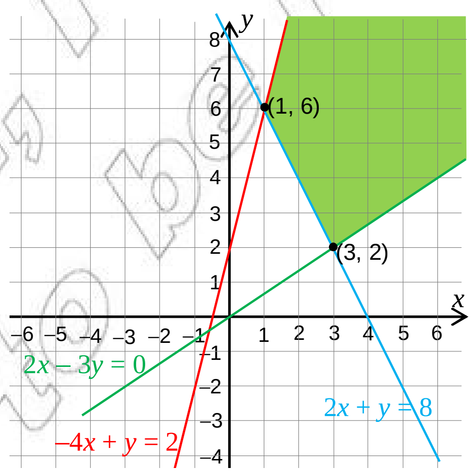


Figure 10.14

Exercise 10.3

FIND THE MAXIMUM AND MINIMUM VALUES OF

A $Z = 2x + 3y$, SUBJECT TO: $x \geq 0$, $y \geq 0$, $2y + x \leq 16$, $x - y \leq 10$

B $Z = 2x + 3y$, SUBJECT TO: $x \geq 0$, $y \geq 0$, $3x + 7y \leq 42$, $x + 5y \leq 22$

C $Z = 4x + 2y$, SUBJECT TO: $x \geq 0$, $y \geq 0$, $x + 2y \geq 4$, $3x + y \geq 7$, $-x + 2y \leq 7$

D $Z = 4x + 5y$ SUBJECT TO: $x \geq 0$, $y \geq 0$, $2x + 2y \leq 10$, $x + 2y \leq 6$

E $Z = 4x + 3y$ SUBJECT TO: $x \geq 0$, $y \geq 0$, $2x + 3y \geq 6$, $3x - 2y \leq 9$, $x + 5y \leq 20$

F $Z = 3x + 4y$ SUBJECT TO: $x \geq 0$, $y \geq 0$, $x + 2y \leq 14$, $3x - y \geq 0$, $x - y \leq 2$

10.4 REAL LIFE LINEAR PROGRAMMING PROBLEMS

Group Work 10.2

1 CONSIDER A FURNITURE SHOP THAT SELLS CHAIR … HERE THE PROFIT PER CHAIR IS BIRR 9 AND THE PROFIT PE… R 7.

A WHAT IS THE PROFIT FROM A SALE OF 6 CHAIRS AN… ABLES?

B IF THE SHOP HAS SOLD x NUMBER OF CHAIRS AND y NUMBERS OF TABLES, WHAT IS THE PROFIT IN TERMS OF x AND y?

2 THE NUMBER OF FIELDS A FARMER PLANTS WITH WHEAT IS W AND THE NUMBER OF FIELDS WITH CORN IS C. THE RESTRICTIONS ON THE NUMBER OF FIELDS ARE THAT:

A THERE MUST BE AT LEAST 2 FIELDS OF CORN.

B THERE MUST BE AT LEAST 2 FIELDS OF WHEAT.

C NOT MORE THAN 10 FIELDS IN TOTAL ARE TO BE SOWN WITH WHEAT OR CORN.

CONSTRUCT THREE INEQUALITIES FROM THE GIVEN INFORMATION AND SKETCH THE REG SATISFIES THE 3 INEQUALITIES.

IN EVERYDAY LIFE, WE ARE OFTEN CONFRONTED WITH A NEED TO ALLOCATE LIMITED RE BEST ADVANTAGE. WE MAY WANT TO MAXIMIZE AN OBJECTIVE FUNCTION (SUCH AS PRO MINIMIZE (SAY, COST) UNDER SOME RESTRICTIONS (WHICH WE CALLED **constraints**)

DESPITE THE APPARENTLY QUITE RESTRICTIVE NATURE OF THE LINEAR PROGRAMMING PROB THERE ARE MANY PRACTICAL PROBLEMS IN INDUSTRY, GOVERNMENT AND OTHER ORGANIZ FALL INTO THIS TYPE. BELOW WE GIVE REAL LIFE EXAMPLES OF SIMPLE LINEAR PROGR PROBLEMS, EACH OF WHICH REPRESENTS A CLASSIC TYPE OF LINEAR PROGRAMMING PROBLE

Example 1 A MANUFACTURER WANTS TO MAXIMIZE THE PROFIT FOR TWO PRODUCTS. PRODUCT I GIVES A PROFIT OF BIRR 1.50 PER KG, AND PRODUCT II GIVES A PROFIT OF BIR 2.00 PER KG. MARKET TESTS AND AVAILABLE RESOURCES HAVE INDICATED T FOLLOWING CONSTRAINTS.

A THE COMBINED PRODUCTION LEVEL SHOULD NOT EXCEED 1200 KG PER MONTH.

B THE DEMAND FOR PRODUCT II IS NOT MORE THAN HALF THE DEMAND FOR PRODUCT I.

C THE PRODUCTION LEVEL OF PRODUCT I IS LESS THAN OR EQUAL TO 600 KG PLUS THREE TIMES THE PRODUCTION LEVEL OF PRODUCT II.

FIND THE NUMBER OF KG OF EACH PRODUCT THAT SHOULD BE PRODUCED IN A MON MAXIMIZE PROFIT.

Solution: THE FIRST STEP IN SOLVING SUCH REAL LIFE LINEAR PROGRAMMING PROBLEMS IS TO ASSIGN VARIABLES TO THE NUMBERS TO BE DETERMINED FOR A MAXIMUM (OR MINIMUM) VALUE OF THE OBJECTIVE FUNCTION.

LET x = THE NUMBER OF KG OF PRODUCT I, AND

y = THE NUMBER OF KG OF PRODUCT II

THESE VARIABLES ARE USUALLY CALLED **decision variables**.

THE OBJECTIVE OF THE MANUFACTURER IS TO DECIDE HOW MANY UNITS OF EACH MU PRODUCED TO MAXIMIZE THE OBJECTIVE FUNCTION (PROFIT) GIVEN BY:

$P = 1.5x + 2y$

THE ABOVE THREE CONSTRAINTS CAN BE TRANSLATED INTO THE FOLLOWING LINEAR INEQUALITIES

A $x + y \le 1200$

B $y \le \frac{1}{2}x$ OR $-x + 2y \le 0$

C $x \le 3y + 600$ OR $x - 3y \le 600$

SINCE NEITHER x NOR y CAN BE NEGATIVE, WE HAVE THE ADDITIONAL NON-NEGATIVITY CONSTRA OF $x \ge 0$ AND $y \ge 0$. THE ABOVE INFORMATION CAN NOW BE TRANSFORMED INTO THE FOLLOW LINEAR PROGRAMMING PROBLEM.

MAXIMIZE $P = 1.5x + 2y$

SUBJECT TO $x \geq 0$

$y \geq 0$

$x + y \leq 1200$

$-x + 2y \leq 0$

$x - 3y \leq 600$

THE CONSTRAINTS ABOVE HAVE REGION OF FEASIBLE SOLUTIONS SHOWN IN FIGURE 10.15

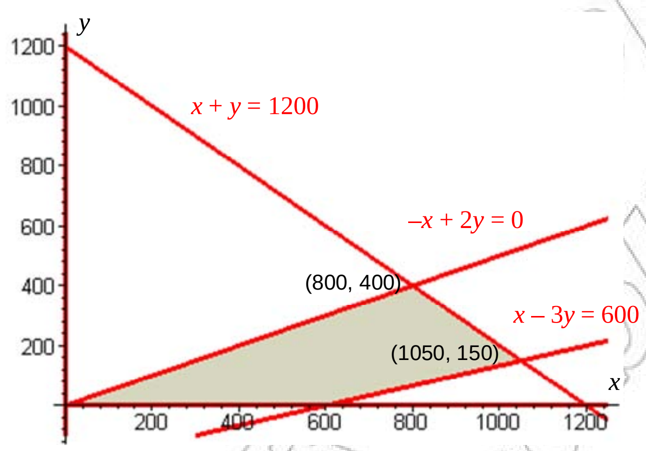


Figure 10.15

TO SOLVE THE MAXIMIZATION PROBLEM GEOMETRICALLY, WE FIRST FIND THE VERTICES B
THE POINTS OF INTERSECTION OF THE BORDER LINES OF S, TO GET

O (0, 0), A (600, 0), B (1050, 150) AND C (800, 400)

THEN A SOLUTION CAN BE OBTAINED FROM THE TABLE BELOW:

Vertex	Profit $P = 1.5x + 2y$
O (0, 0)	P = 1.5 (0) + 2(0) = 0
A (600, 0)	P = 1.5 (600) + 2(0) = 900
B (1050, 150)	P = 1.5 (1050) + 2 (150) = 1875
C (800, 400)	P = 1.5 (800) + 2 (400) = 2000

THUS THE MAXIMUM PROFIT IS BIRR 2000 AND IT OCCURS WHEN THE MONTHLY PRODU
CONSISTS OF 800 UNITS OF PRODUCT I AND 400 UNITS OF PRODUCT II.

(OBSERVE THAT THE MINIMUM PROFIT IS BIRR 0 WHICH OCCURS AT THE VERTEX O (0, 0)).

Example 2 A MANUFACTURER OF TENTS MAKES A STANDARD MODEL AND AN EXPEDITION M
FOR NATIONAL DISTRIBUTION. EACH STANDARD TENT REQUIRES 1 LABOUR-HO

THE CUTTING DEPARTMENT AND 3 LABOUR-HOURS FROM THE ASSEMBLY DEPAR
EACH EXPEDITION TENT REQUIRES 2 LABOUR-HOURS FROM CUTTING AND 4 LA
HOURS FROM ASSEMBLY. THE MAXIMUM LABOUR-HOURS AVAILABLE PER DA
THE CUTTING DEPARTMENT AND THE ASSEMBLY DEPARTMENT ARE 32 AND
RESPECTIVELY. IF THE COMPANY MAKES A PROFIT OF BIRR 50.00 ON EACH
STANDARD TENT AND BIRR 80 ON EACH EXPEDITION TENT, HOW MANY TENTS OF
TYPE SHOULD BE MANUFACTURED EACH DAY TO MAXIMIZE THE TOTAL DAILY P
(ASSUME THAT ALL TENTS PRODUCED CAN BE SOLD.)

Solution: THE INFORMATION GIVEN IN THE PROBLEM CAN BE SUMMARIZED IN THE FOLLOWING TABLE.

	Labour-hr per tent		Max. Labour-hr per day
	Standard	**Expedition**	
Cutting dept	1	2	32
Assembly dept	3	4	84
Profit	BIRR 50	BIRR 80	

THEN WE ASSIGN DECISION VARIABLES AS FOLLOWS:

LET x = NUMBER OF STANDARD TENTS PRODUCED PER DAY

y = NUMBER OF EXPEDITION TENTS PRODUCED PER DAY

THE OBJECTIVE OF MANAGEMENT IS TO DECIDE HOW MANY OF EACH TENT SHOULD BE PRODUCED EACH DAY IN ORDER TO MAXIMIZE PROFIT P = $50x + 80y$

BOTH CUTTING AND ASSEMBLY DEPARTMENTS HAVE THE CONSTRAINTS GIVEN BY

$1\times x + 2\times y \leq 32$ . . . . .CUTTING DEPT. CONSTRAINT

$3\times x + 4\times y \leq 84$ . . . . ASSEMBLY DEPT. CONSTRAINT

WHERE $x \geq 0$ AND $y \geq 0$ . . . . NON-NEGATIVE CONSTRAINTS

THE LINEAR PROGRAMMING PROBLEM IS THEN TO MAXIMIZE P = $50x + 80y$,

SUBJECT TO: $x + 2y \leq 32$

$3x + 4y \leq 84$

$x, y \geq 0$

TO GET A GRAPHICAL SOLUTION, WE HAVE THE FEASIBLE REGION SHOWN IN FIGURE 10.16. THE VERTICES ARE AT (0, 0), (28, 0), (20, 6) AND (0, 16). THE MAXIMUM VALUE OF PROFIT CAN BE OBTAINED FROM THE FOLLOWING TABLE.

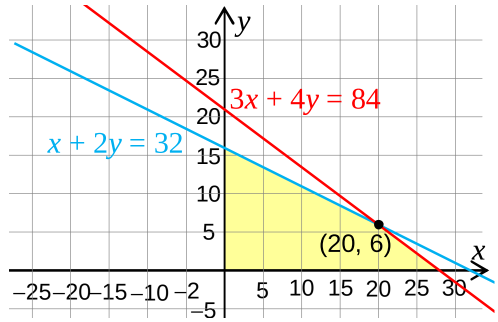


Figure 10.16

Vertex	Value of P = 50x + 80y
(0, 0)	$P = 50(0) + 80(0) = 0$
(28, 0)	$P = 50\ (28) + 80\ (0) = 1{,}400$
(20, 6)	$P = 50\ (20) + 80\ (6) = 1{,}480$
(0, 16)	$P = 50\ (0) + 80\ (16) = 1{,}280$

THUS THE MAXIMUM PROFIT OF BIRR 1,480 IS ATTAINED AT (20, 6); I.E. THE MANUFACTURER SHOULD PRODUCE 20 STANDARD AND 6 EXPEDITION TENTS EACH DAY TO MAXIMIZE PROFIT.

Example 3 A PATIENT IN A HOSPITAL IS REQUIRED TO HAVE AT LEAST 84 UNITS OF DRUG A 120 UNITS OF DRUG B EACH DAY. EACH GRAM OF SUBSTANCE M CONTAINS 10 UN OF DRUG A AND 8 UNITS OF DRUG B, AND EACH GRAM OF SUBSTANCE N CONTAI UNITS OF DRUG A AND 4 UNITS OF DRUG B. SUPPOSE BOTH SUBSTANCES M AND CONTAIN AN UNDESIRABLE DRUG C, 3 UNITS PER GRAM IN M AND 1 UNIT PER GR IN N. HOW MANY GRAMS OF EACH SUBSTANCE M AND N SHOULD BE MIXED TO MEET THE MINIMUM DAILY REQUIREMENTS AND AT THE SAME TIME MINIMIZE T INTAKE OF DRUG C? HOW MANY UNITS OF DRUG C WILL BE IN THIS MIXTURE?

Solution LET US SUMMARIZE THE ABOVE INFORMATION AS:

	Substance M	Substance N	Min-requirement
Drug A	10	2	84
Drug B	8	4	120
Drug C	3	1	

LET x = NUMBER OF GRAMS OF SUBSTANCE M USED

y = NUMBER OF GRAMS OF SUBSTANCE N USED

OUR OBJECTIVE IS TO MINIMIZE DRUG C: FROM $3x + y$.

THE CONSTRAINTS ARE THE MINIMUM REQUIREMENTS OF

$10x + 2y \geq 84$ . . . . FROM DRUG A

AND $8x + 4y \geq 120$ . . . . FROM DRUG B

SINCE BOTH M AND N MUST BE NON NEGATIVE $x \ge 0, y \ge 0$.

THUS OUR OPTIMIZATION PROBLEM IS TO

MINIMIZE $C = 3x + y$,

SUBJECT TO $10x + 2y \ge 84$

$8x + 4y \ge 120$

$x, y \ge 0$

THE SKETCH OF THE FEASIBLE REGION IS GIVEN IN FIGURE 10.17.

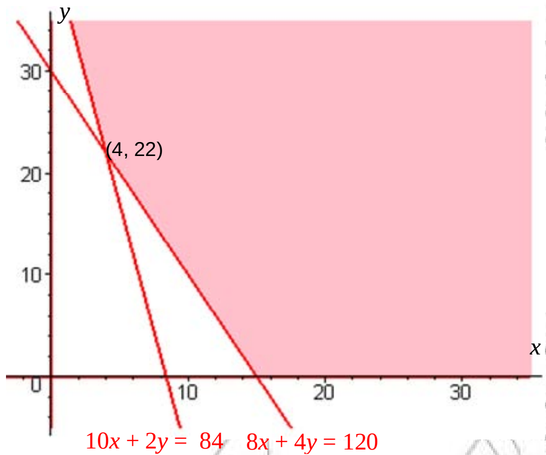


Figure 10.17

TO OBTAIN THE MINIMUM VALUE GRAPHICALLY, WE USE THE TABLE

Vertex	Value of C = 3x + y
(0, 42)	C = 3(0) + 42 = 42
(4, 22)	C = 3 (4) + 22 = 34
(15, 0)	C = 3(15) + 0 = 45

THE MINIMUM INTAKE OF DRUG C IS 34 UNITS AND IT IS ATTAINED AT AN INTAKE OF 4 GRAMS SUBSTANCE M AND 22 GRAMS OF SUBSTANCE N.

WE CAN SUMMARIZE THE STEPS IN SOLVING REAL LIFE OPTIMIZATION PROBLEMS GEOMETR FOLLOWS.

Step 1: SUMMARIZE THE RELEVANT INFORMATION IN THE PROBLEM IN TABLE FORM.

Step 2: FORM A MATHEMATICAL MODEL OF THE PROBLEM BY INTRODUCING DECISION VARIA AND EXPRESSING THE OBJECTIVE FUNCTION AND THE CONSTRAINTS USING THESE VAR

step 3: GRAPH THE FEASIBLE REGION AND FIND THE CORNER POINTS.

Step 4: CONSTRUCT A TABLE OF THE VALUES OF THE OBJECTIVE FUNCTION AT EACH VERTE

Step 5: DETERMINE THE OPTIMAL VALUE(S) FROM THE TABLE.

Step 6: INTERPRET THE OPTIMAL SOLUTION(S) IN TERMS OF THE ORIGINAL REAL LIFE PROBL

Exercise 10.4

SOLVE EACH OF THE FOLLOWING REAL LIFE PROBLEMS:

A A FARMER HAS BIRR 1,700 TO BUY SHEEP AND GOATS. SUPPOSE THE UNIT PRICE O SHEEP IS BIRR 300 AND THE UNIT PRICE OF GOATS IS BIRR 200.

I IF HE DECIDED TO BUY ONLY GOATS, WHAT IS THE MAXIMUM NUMBER OF GO HE CAN BUY?

II IF HE HAS BOUGHT 2 SHEEP WHAT IS THE MAXIMUM NUMBER OF GOATS HE CA BUY WITH THE REMAINING MONEY?

III CAN THE FARMER BUY 4 SHEEP AND 3 GOATS? 2 SHEEP AND 5 GOATS? 3 SHEEP AND 4 GOATS?

B A COMPANY PRODUCES TWO TYPES OF TABLES; TABLES A AND TABLE B. IT TAKES HOURS OF CUTTING TIME AND 4 HOURS OF ASSEMBLING TO PRODUCE TABLE A. IT T 10 HOURS OF CUTTING TIME AND 3 HOURS OF ASSEMBLING TO PRODUCE TABLE B. COMPANY HAS AT MOST 112 HOURS OF CUTTING LABOUR AND 54 HOURS OF ASSEM LABOUR PER WEEK THE COMPANY'S PROFIT IS BIRR 60 FOR EACH TABLE A PRODUC AND BIRR 170 FOR EACH TABLE B PRODUCED. HOW MANY OF EACH TYPE OF TAB SHOULD THE COMPANY PRODUCE IN ORDER TO MAXIMIZE PROFIT?

C THE OFFICERS OF A HIGH SCHOOL SENIOR CLASS ARE PLANNING TO RENT BUSES AN FOR A CLASS TRIP. EACH BUS CAN TRANSPORT 36 STUDENTS, REQUIRES 4 SUPERVISO COSTS BIRR 1000 TO RENT. EACH VAN CAN TRANSPORT 6 STUDENTS, REQUIRE SUPERVISOR, AND COSTS BIRR150 TO RENT. THE OFFICERS MUST PLAN TO ACCOMM AT LEAST 420 STUDENTS. SINCE ONLY 48 PARENTS HAVE VOLUNTEERED TO SER SUPERVISORS, THE OFFICERS MUST PLAN TO USE AT MOST 48 SUPERVISORS. HOW M VEHICLES OF EACH TYPE SHOULD THE OFFICERS RENT IN ORDER TO MINIMIZE TRANSPORTATION COSTS? WHAT IS THE MINIMUM TRANSPORTATION COST?

Key Terms

bounded solution region	**minimum value**
constraints	**objective function**
decision variables	**optimal value**
equation of a line	**real life linear programming problems**
Fundamental theorem of linear programming	**slope of a line**
half planes	**solution region**
inclination of a line	**system of linear inequalities**
maximum value	**vertex (corner point)**

Summary

1. THE **angle of inclination** OF A LINE L IS THE ANGLE MEASURED FROM THE x-AXIS TO L IN THE COUNTER CLOCKWISE DIRECTION.
2. THE **slope** OF A LINE PASSING THROUGH P(x_1, y_1) AND Q(x_2, y_2) IS

$$m = \text{TAN} = \frac{y_2 - y_1}{x_2 - x_1},\ for\ x_1 \neq x_2.$$

3. IF A LINE HAS SLOPE m AND PASSES THROUGH P(x_1, y_1), THE SLOPE-POINT FORM OF ITS **equation** IS GIVEN BY $y - y_1 = m(x - x_1)$
4. AN EQUATION OF A LINE CAN BE REDUCED TO THE FORM $ax + by = c$, $a, b, c \in \mathbb{R}$ WITH $a \neq 0$ OR $b \neq 0$.
5. A LINE ℓ DIVIDES THE PLANE INTO TWO **half-planes.**
6. A **system of linear inequalities** IS A COLLECTION OF TWO OR MORE LINEAR INEQUALITIES TO BE SOLVED SIMULTANEOUSLY.
7. A **graphical solution** IS THE COLLECTION OF ALL POINTS THAT SATISFY THE SYSTEM OF LINEAR INEQUALITIES.
8. A **vertex** (OR **corner point**) OF A SOLUTION REGION IS A POINT OF INTERSECTION OF TWO OR MORE BOUNDARY LINES.
9. A SOLUTION REGION IS SAID TO BE **bounded**, IF IT CAN BE ENCLOSED IN A RECTANGLE.
10. A NUMBER $M = f(c)$ FOR c IN I IS CALLED THE **maximum value** OF f ON I, IF $M \geq f(x)$ FOR ALL x IN I.
11. A NUMBER $m = f(d)$ FOR d IN I IS CALLED THE **minimum value** OF f ON I, IF $m \leq f(x)$ FOR ALL x IN I.
12. A VALUE WHICH IS EITHER A MAXIMUM OR A MINIMUM VALUE IS CALLED AN **optimal** (OR **extremum**) VALUE.
13. AN OPTIMIZATION PROBLEM INVOLVES MAXIMIZING OR MINIMIZING AN **objective function** SUBJECT TO **constraints**.
14. IF AN OPTIMAL VALUE OF AN OBJECTIVE FUNCTION EXISTS, IT WILL OCCUR AT ONE OR THE CORNER POINTS OF THE FEASIBLE REGION.
15. IN SOLVING REAL LIFE LINEAR PROGRAMMING PROBLEMS, ASSIGN VARIABLES CALLED **decision variables.**

Review Exercises on Unit 10

1. FIND THE SLOPE OF THE LINE
   - **A** THAT PASSES THROUGH THE POINTS P(−1, 0) AND Q(4, 2)
   - **B** THAT HAS ANGLE OF INCLINATION $= 45^{o}$

C THAT IS PARALLEL TO THE LINE $2x - 4y = 6$

2 DRAW THE GRAPHS OF THE LINES ℓ_1: $y = 3x - 4$ AND ℓ_2: $x - 5y = 2$ USING THE SAME COORDINATE AXES.

3 FIND GRAPHICAL SOLUTIONS FOR EACH OF THE FOLLOWING SYSTEMS OF LINEAR INEQUALITIES.

A $x - 5y \le 2$
$3x - y \le 4$

B $y + 2x \ge 4$
$y - 2x > 4$

C $x \ge 2$
$y \ge 0$
$x + y \le 5$

D $x \ge 0$
$y \ge 0$
$3x + 2y < 6$

4 FIND THE MAXIMUM AND MINIMUM VALUES OF THE OBJECTIVE FUNCTION SUBJECT TO THE GIVEN CONSTRAINTS.

A OBJECTIVE FUNCTION $Z = 3x + 2y$,
SUBJECT TO: $x \ge 0$
$y \ge 0$
$x + 3y \le 15$
$4x + y \le 16$

B OBJECT FUNCTION $Z = 2x + 3y$,
SUBJECT TO: $x \ge 0$
$y \ge 0$
$2x + y \ge 100$
$x + 2y \ge 80$

C OBJECTIVE FUNCTION $Z = 10x + 7y$
SUBJECT TO: $0 \le x \le 60$
$0 \le y \le 45$
$5x + 6y \le 420$

D OBJECTIVE FUNCTION $Z = 3x + 4y$,
SUBJECT TO: $x \ge 1$,
$y \ge 0$
$3x - 4y \le 12$
$x + 2y \ge 4$

5 FIND THE OPTIMAL SOLUTION OF THE FOLLOWING LINEAR PROGRAMMING PROBLEMS.

A AHADU COMPANY PRODUCES TWO MODELS OF RADIOS. MODEL A REQUIRES 20 MIN OF WORK ON ASSEMBLY LINE I AND 10 MIN OF WORK ON ASSEMBLY LINE II. MODEL B REQUIRES 10 MIN OF WORK ON ASSEMBLY LINE I AND 15 MIN OF WORK ON ASSEMBLY LINE II. AT MOST 22 HRS OF ASSEMBLY TIME ON LINE I AND 25 HRS OF ASSEMBLY TIME ON LINE II ARE AVAILABLE PER WEEK IT IS ANTICIPATED THAT AHADU COMPANY WILL REALIZE A PROFIT OF BIRR 10 ON MODEL A AND BIRR 14 ON MODEL B. HOW MANY RADIOS OF EACH MODEL SHOULD BE PRODUCED PER WEEK IN ORDER TO MAXIMIZE AHADU'S PROFIT?

B A FARMING COOPERATIVE MIXES TWO BRANDS OF CATTLE FEED. BRAND X COSTS BIRR 25 PER BAG AND CONTAINS 2 UNITS OF NUTRITIONAL ELEMENT A, 2 UNITS OF NUTRITIONAL ELEMENT B, AND 2 UNITS OF ELEMENT C. BRAND Y COSTS BIRR 20 PER BAG AND CONTAINS 1 UNIT OF NUTRITIONAL ELEMENT A, 9 UNITS OF ELEMENT B, AND 3 UNITS OF ELEMENT C. THE MINIMUM REQUIREMENTS OF NUTRIENTS A, B AND C ARE 12, 36 AND 24 UNITS, RESPECTIVELY. FIND THE NUMBER OF BAGS OF EACH BRAND THAT SHOULD BE MIXED TO PRODUCE A MIXTURE HAVING A MINIMUM COST.